

Renormalized solution to a class of elliptic equations in a two-component domain with nonlinear imperfect transmission

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ABSTRACT

In this article, we prove the existence and uniqueness of a renormalized solution of a class of elliptic problems posed in a two-component domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ with L^1 data. On the interface, we prescribe that the jump of the solution is related to the flux nonlinearly.

Existence is proved by a classical approximation argument: we introduce an approximate problem and show that its solutions converge to a renormalized solution. The uniqueness result requires certain strict monotonicity conditions and an additional interface regularity lemma

INTRODUCTION

In this work, we study renormalized solution to a class of elliptic problems set in a two-component domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ with nonlinear imperfect transmission on the interface. More specifically, we study the problem that was considered by Donato and Le Nguyen (2015), which models thermal conduction in a two-component domain with nonlinear transmission on the interface. However, in place of square-integrable data f , we consider an integrable function f , that is, $f \in L^1(\Omega)$. As such, the two components of Ω , denoted by Ω_1 and Ω_2 , are disjoint open subsets of Ω that share a common boundary, denoted by the interface Γ . The open sets Ω_1 and Ω_2 model the region occupied by two materials interacting through the interface Γ as described by Auriault and Ene (1994), Carslaw and Jaeger (1959), and Hummel (2000). In this model, the rate of heat exchanged between the two materials through the interface depends nonlinearly on the jump on the interface.

To be precise, we analyze the following class of elliptic problems:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_1) + g_1(x, u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(A(x)\nabla u_2) + g_2(x, u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (A(x)\nabla u_1)v_1 = -(A(x)\nabla u_2)v_2 & \text{on } \Gamma, \\ (A(x)\nabla u_1)v_1 = -h(x, u_1 - u_2) & \text{on } \Gamma. \end{cases} \quad (1)$$

The functions h , g_1 and g_2 are nondecreasing Carathéodory functions that capture the nonlinearity of the problem. The function f is in $L^1(\Omega)$, while the matrix field A , which represents the conductivity of the material, is assumed to be bounded and elliptic. Unlike classical domains, problems set in two-component domains require an interface coupling, represented by the function h , that models the interaction between the solutions in the two-components.

The nonlinear term h is motivated by the fact that in some experimental results, the interface flux is nonlinearly related to the solution jump (one may refer to (Carslaw and Jaeger 1959)).

Problem (1) was previously studied in (Monsurro 2003; Monsurro (Erratum) 2003) by studying its weak solutions and homogenization in its linear form with g_1 and g_2 equal to zero and h depending on x only.

Since the datum belongs only to $L^1(\Omega)$, the existence of a weak solution is not guaranteed. We therefore work within a different solution framework suitable for such low-regularity data.

There are various frameworks that can be used for weak data (e.g., L^1 data, measure data). Relevant frameworks include the following: solutions obtained as limit approximations (SOLA) - introduced by Dall'Aglio in (2021); entropy solutions - introduced by Benilan et al. in (1995) and renormalized solutions - introduced by DiPerna and Lions (1989).

DiPerna and Lions (1989), the authors showed the existence and uniqueness of renormalized solutions to the Boltzmann equation, which is a first-order PDE. Murat and Lions in (Murat 1991; Lions and Murat. Sur les solutions renormalisées d'équations elliptiques. (unpublished manuscript)) paved the way for the adaptation of the

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KEYWORDS

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renormalized solution for second-order problems. To date, there is an extensive literature that present existence and uniqueness of renormalized solutions of different problems. Examples include (Dal Maso et al. 1999; Betta et al. 2015; Betta et al. 2019; Fulgencio and Guibé 2021a, 2021b; Guibé and Oropeza 2017).

Fulgencio and Guibé 2021a, 2021b) studied the quasilinear version of (1) was studied where $g_i \equiv 0$, $i = 1, 2$ and the function h is linear with respect to the second variable. The definition of the renormalized solution of (1) (see Definition 5) is inspired by the definition in (Fulgencio and Guibé 2021a, 2021b). In particular, the additional trace condition of the solution are adapted from analogous conditions in (Fulgencio and Guibé 2021a).

The main difficulty here is dealing with the nonlinear terms both in the interface integral and in the integrals over the two components, Ω_1 and Ω_2 . Moreover, when proving uniqueness, we need to prove that a solution is in $L^1(\Omega)$ and its trace is in $L^1(\Gamma)$, to justify passage to the limit in the relevant integrals. This will be done by using the ideas from Boccardo and Gallouët (1989). Moreover, we also need to assume that at least one of g_2 and h in (1) is strictly increasing.

This article is organized as follows. In the next section, we discuss some preliminaries. Here, we present the assumptions that we impose on the problem and the appropriate function space. The third section is dedicated to our existence and uniqueness results. We first consider an approximate problem to prove the existence of a solution. For uniqueness, we establish an interface regularity lemma.

PRELIMINARIES

In a given domain $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$, consider an open subset $\Omega_2 \subset \Omega$ such that $\overline{\Omega_2}$ is properly contained in Ω (see Figure 1). We define $\Omega_1 \doteq \Omega \setminus \overline{\Omega_2}$ and $\Gamma \doteq \partial\Omega_2$. Given a function u defined almost everywhere on Ω , we denote by u_i the restriction of u on Ω_i for $i = 1, 2$, that is, $u_i = u|_{\Omega_i}$.

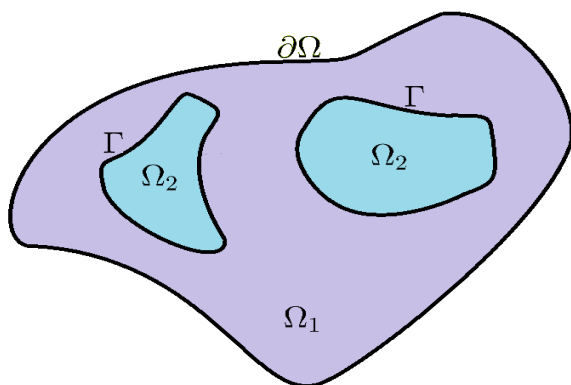


Figure 1: The domain Ω with its components Ω_1 (purple) and Ω_2 (blue).

Notice that, due to the nature of a solution to (1), we need to introduce a function space adapted to the transmission problem. We define the following Sobolev space V :

$$V = \{(u_1, u_2) : u_1 \in V_1 \text{ and } u_2 \in H^1(\Omega_2)\}$$

where $V_1 = \{w \in H^1(\Omega_1) : w = 0 \text{ on } \partial\Omega\}$. We equip V with the following norm

$$\|u\|_V^2 = \|\nabla u_1\|_{L^2(\Omega_1)}^2 + \|\nabla u_2\|_{L^2(\Omega_2)}^2 + \|u_1 - u_2\|_{L^2(\Gamma)}^2.$$

It may be shown to have an equivalent norm according to the proposition below.

Proposition 1. (Monsurro 2003) *There exists a constant $c_1, c_2 > 0$ such that*

$$c_1 \|u\|_V \leq \|u\|_{V_1 \times H^1(\Omega_2)} \leq c_2 \|u\|_V.$$

The properties imposed to the given functions in (1) are listed below.

Definition 1. A function $f(x, y)$ on $\mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be Carathéodory if the following condition holds:

1. $f(\cdot, y)$ is measurable for every $y \in \mathbb{R}$; and
2. $f(x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^N$.

Definition 2 (Assumption $\mathcal{H}_1$). A function $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy $\mathcal{H}_1$ if g is a Carathéodory function such that $g(x, \cdot)$ is a nondecreasing function in $C^1(\mathbb{R})$ with $g(x, 0) = 0$ for almost all $x \in \mathbb{R}^N$.

Definition 3 (Assumption $\mathcal{H}_2$). A function $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy $\mathcal{H}_2$ if there is a constant $B_g > 0$ such that

$$\left| \frac{\partial g}{\partial t}(x, t) \right| \leq B_g \text{ for almost every } x \in \Omega \text{ and for all } t \in \mathbb{R}.$$

Remark 2. (Donato and Le Nguyen 2015) *Let g be a function that satisfies $\mathcal{H}_2$ and let B_g be a positive constant as in Assumption $\mathcal{H}_2$. Then, for all $x \in \Omega$ and $s \in \mathbb{R}$,*

$$|g(x, s)| = |g(x, s) - g(x, 0)| = \left| \int_0^s \frac{\partial g}{\partial t}(x, t) dt \right| \leq B_g |s|.$$

Moreover,

$$|g(x, b) - g(x, a)| = \left| \int_a^b \frac{\partial g}{\partial t}(x, t) dt \right| \leq B_g (b - a).$$

Definition 4 (Assumption $\mathcal{H}_3$). A function $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy $\mathcal{H}_3$ if there exists a constant $C > 0$ such that $tg(x, t) \geq C|t|^2$ for almost every $x \in \Omega$ and for all $t \in \mathbb{R}$.

We then impose the following assumptions for problem (1).

- (A1) The function f is in $L^1(\Omega)$.
- (A2) The matrix field $A: \Omega \rightarrow \mathbb{R}^{N \times N}$ is in $M(\alpha, \beta, \Omega)$, that is,
 - a. $\alpha|\xi|^2 \leq A\xi \cdot \xi$ for all $\xi \in \mathbb{R}^N$;
 - b. $|A\xi| \leq \beta|\xi|$ for all $\xi \in \mathbb{R}^N$.
- (A3) The functions g_1 and g_2 satisfy Assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$.
- (A4) The function h satisfies Assumptions $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$.

Remark 3. *Assumption $\mathcal{H}_2$ in this work is a special case of assumptions $\mathcal{H}_2$, and $\mathcal{H}_4$ in Donato and Le Nguyen (2015).*

A weak solution to (1) may not exist if we assume that $f \in L^1(\Omega)$ instead of $f \in L^2(\Omega)$. Hence, we use the renormalized framework introduced by DiPerna and Lions (1989) and extended to second-order problems by Murat and Lions (1991). We first define the truncation function is given by

$$T_k(s) = \begin{cases} k & \text{for } s \geq k \\ s & \text{for } -k < s < k \\ -k & \text{for } s \leq -k. \end{cases}$$

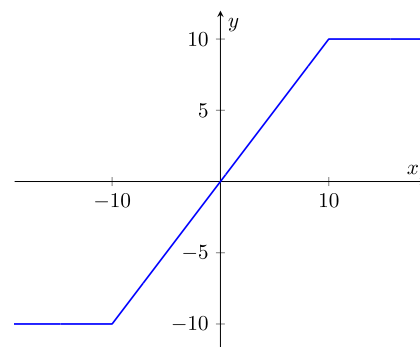


Figure 2: The truncation function of height 10.

Remark 4. Let $s, t \in \mathbb{R}$ with $s \neq t$.

1. Since T_k is nondecreasing,

$$\frac{T_k(s) - T_k(t)}{s - t} \geq 0.$$
2. Also T_k is Lipschitz continuous with constant 1, that is,
 $|T_k(s) - T_k(t)| \leq |s - t|.$
Then,

$$(T_k(s) - T_k(t))^2 \leq (T_k(s) - T_k(t))(s - t). \quad (2)$$
3. If g is a function satisfying $\mathcal{H}_2$, then for all $x \in \mathbb{R}^N$,

$$|(T_k(s) - T_k(t))(g(x, s) - g(x, t))| \leq C|T_k(s) - T_k(t)||s - t|.$$
4. If g is a function satisfying $\mathcal{H}_3$, then there exists a constant $g_0 > 0$ such that for all $x \in \mathbb{R}^N$,

$$(T_k(s) - T_k(t))g(x, s - t) = \left(\frac{T_k(s) - T_k(t)}{s - t}\right)(s - t)g(x, s - t) \geq \left(\frac{T_k(s) - T_k(t)}{s - t}\right)(g_0)(s - t)^2 = g_0(T_k(s) - T_k(t))(s - t). \quad (3)$$
5. If $g(x, t)$ is nondecreasing with respect to the second variable, then for all $x \in \mathbb{R}^N$,

$$(T_k(s) - T_k(t))(g(x, s) - g(x, t)) \geq 0. \quad (4)$$

In particular, if $g(x, 0) = 0$, for a.e. $x \in \mathbb{R}^N$, then
 $sg(x, s) \geq 0.$

The properties above, except property 2, also hold if we replace T_k by any nondecreasing function in $\mathbb{R}$.

. Because a solution of (1) is not known a priori to belong to a Sobolev space, we define its gradient and trace through the gradients and traces of its truncations. In particular, we have the following proposition, which was proved by Fulgencio and Guibé (2021a) with the proof inspired by Bénilan et al. (1995) and Guibé and Oropeza (2017).

Proposition 5 (Bénilan et al. 1995; Guibé and Oropeza 2017; Fulgencio and Guibé 2021a). Let $u = (u_1, u_2): \Omega \setminus \Gamma \rightarrow \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k > 0$:

1. For $i = 1, 2$, there exists a measurable function $G_i: \Omega_i \rightarrow \mathbb{R}^N$ such that for all $k > 0$,

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i,$$
where $\chi_{\{|u_i| < k\}}$ denotes the characteristic function of the set $\{x \in \Omega_i: |u_i(x)| < k\}$. The function G_i is defined to be the gradient of u_i and we write $G_i = \nabla u_i$.

2. Suppose

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty.$$

Then there exists a measurable function $w_i: \Gamma \rightarrow \mathbb{R}$, for $i = 1, 2$, such that for all $k > 0$,
 $\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma$, where $\gamma_i: H^1(\Omega) \rightarrow L^2(\Gamma)$ is the trace operator. The trace of u_i is defined to be the function w_i and we write $\gamma_i(u_i) = w_i$.

We now have the definition of a renormalized solution of (1).

Definition 5. Let $u = (u_1, u_2): \Omega \setminus \Gamma \rightarrow \mathbb{R}$ be a measurable function. Then, u is a renormalized solution of (1) if

$$T_k(u) \in V, \quad \text{for all } k > 0; \quad (5)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \text{for all } k > 0; \quad (6)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u| < k\}} A(x) \nabla u \nabla u \, dx = 0; \quad (7)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\Gamma} (u_1 - u_2)(T_k(u_1) - T_k(u_2)) \, d\sigma = 0; \quad (8)$$

and for all $\psi_1, \psi_2 \in C^1(\mathbb{R})$ (equivalently, $\psi_1, \psi_2 \in W^{1,\infty}(\mathbb{R})$) with compact support,

$$\begin{aligned} & \int_{\Omega_1} \psi_1(u_1) A(x) \nabla u_1 \nabla \phi_1 \, dx + \int_{\Omega_2} \psi_2(u_2) A(x) \nabla u_2 \nabla \phi_2 \, dx \\ & + \int_{\Omega_1} \phi_1 \psi_1'(u_1) A(x) \nabla u_1 \nabla u_1 \, dx + \int_{\Omega_2} \phi_2 \psi_2'(u_2) A(x) \nabla u_2 \nabla u_2 \, dx \\ & + \int_{\Omega_1} \psi_1(u_1) g_1(x, u_1) \phi_1 \, dx + \int_{\Omega_2} \psi_2(u_2) g_2(x, u_2) \phi_2 \, dx \\ & + \int_{\Gamma} h(x, u_1 - u_2) (\phi_1 \psi_1(u_1) - \phi_2 \psi_2(u_2)) \, d\sigma \\ & = \int_{\Omega_1} f \phi_1 \psi_1(u_1) \, dx + \int_{\Omega_2} f \phi_2 \psi_2(u_2) \, dx \end{aligned} \quad (9)$$

for $\phi = (\phi_1, \phi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

Remark 6. The proof that the first four integrals and the last two integrals in (9) are well-defined is very similar to that of in Fulgencio and Guibé (2021a). We show that the remaining integrals in (9) are also well defined.

Let $\psi_1, \psi_2 \in C^1(\mathbb{R})$ with compact support and $\phi = (\phi_1, \phi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$. For the integrals involving g_1 and g_2 , by Remark 2, for $i = 1, 2$,

$$|\psi_i(u_i) g_i(x, u_i) \phi_i| = |\psi_i(u_i) g_i(x, T_n(u_i)) \phi_i| \leq Cn \|\psi\|_{L^\infty(\mathbb{R})} \|\phi_i\|_{L^\infty(\Omega_i)},$$

where for some $n > 0$, $\text{supp } \psi_i \subseteq [-n, n]$. Hence, $\psi_i(u_i) g_i(x, u_i) \phi_i$ is integrable over Ω_i for all $\phi_i \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

For the integral on the interface, for $k > 0$, define the continuous function $S_k: \mathbb{R} \rightarrow \mathbb{R}$ by

$$S_k(t) = \begin{cases} 0, & \text{if } t > 2k \\ -\frac{t}{k} + 2, & \text{if } k < t \leq 2k \\ 1, & \text{if } -k \leq t \leq k \\ \frac{t}{k} + 2, & \text{if } -2k \leq t < -k \\ 0, & \text{if } t < -2k. \end{cases} \quad (10)$$

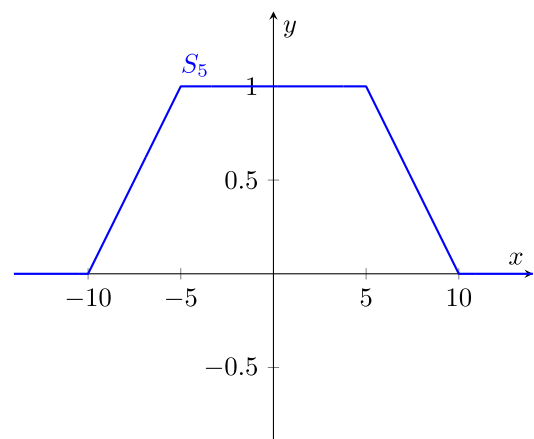


Figure 3: The graph of the function S_5 .

Observe that since $\text{supp}\psi_i \subseteq [-n, n]$,

$$\begin{aligned} & h(x, u_1 - u_2) \phi_1 \psi_1(u_1) \\ &= h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_1) \\ &= h(x, u_1 - u_2) \phi_1 (\psi_1(u_1) - \psi_1(u_2)) S_n(u_1) \\ &+ h(x, u_1 - u_2) \phi_1 \psi_1(u_2) S_n(u_1). \end{aligned} \quad (11)$$

Noting that ψ_1 is a $C^1(\mathbb{R})$ -function with compact support, we have $|\psi_1(s) - \psi_1(t)| \leq \|\psi_1'\|_{L^\infty(\mathbb{R})} |s - t|$.

Also, because ψ_1 has compact support contained in $[-n, n]$, it follows that

$$|\psi_1(u_1) - \psi_1(u_2)| = |\psi_1(T_n(u_1)) - \psi_1(T_n(u_2))| \leq \|\psi_1'\|_{L^\infty(\mathbb{R})} |T_n(u_1) - T_n(u_2)|.$$

Since h satisfies Assumption $\mathcal{H}_2$, there is $C > 0$ such that

$$\begin{aligned} & h(x, u_1 - u_2) \phi_1 (\psi_1(u_1) - \psi_1(u_2)) S_n(u_1) \\ & \leq C |u_1 - u_2| \|\phi_1\|_{L^\infty(\Gamma)} \|\psi_1'\|_{L^\infty(\mathbb{R})} |T_{2n}(u_1) - T_{2n}(u_2)|. \end{aligned}$$

By (6), the term on the right-hand side of the preceding inequality is in $L^1(\Gamma)$. Hence, $h(x, u_1 - u_2) \phi_1 (\psi_1(u_1) - \psi_1(u_2)) S_n(u_1)$ is integrable. Observe that for the other term in (11), by noting that $\text{supp}\psi_i \subseteq [-n, n]$ and $\text{supp}S_k \subseteq [-2k, 2k]$, for any $k > 0$, we have by Remark 2,

$$\begin{aligned} |h(x, u_1 - u_2) \phi_1 \psi_1(u_2) S_n(u_1)| &= |h(x, T_{2n}(u_1) - T_n(u_2)) \phi_1 \psi_1(u_2) S_n(u_1)| \\ &\leq B_h |T_{2n}(u_1) - T_n(u_2)| \|\phi_1\|_{L^\infty(\Omega_1)} \|\psi_1\|_{L^\infty(\mathbb{R})} \\ &\leq B_h(3n) \|\phi_1\|_{L^\infty(\Omega_1)} \|\psi_1\|_{L^\infty(\mathbb{R})} \in L^1(\Gamma). \end{aligned}$$

Since $h(x, u_1 - u_2) \phi_1 \psi_1(u_1)$ is a sum of $L^1(\Gamma)$ functions, we conclude that it is integrable. A similar argument to conclude the same result for $h(x, u_1 - u_2) \phi_2 \psi_2(u_2)$. Thus, the interface integral is well defined.

MAIN RESULTS

We begin this section with a few lemmas needed to prove existence of a renormalized solution of problem (1). After proving existence, we establish uniqueness.

Let $f^\varepsilon \in L^2(\Omega)$ be a sequence of functions that converges strongly to the function f in $L^1(\Omega)$. We begin by studying the approximate problem:

$$\begin{cases} \text{Find } u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in V \text{ such that} \\ \int_{\Omega_1} A(x) \nabla u_1^\varepsilon \nabla \phi_1 dx + \int_{\Omega_1} A(x) \nabla u_2^\varepsilon \nabla \phi_2 dx \\ + \int_{\Omega_1} g_1(x, u_1^\varepsilon) \phi_1 dx + \int_{\Omega_2} g_2(x, u_2^\varepsilon) \phi_2 dx \\ + \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) (\phi_1 - \phi_2) d\sigma \\ = \int_{\Omega_1} f^\varepsilon \phi_1 dx + \int_{\Omega_2} f^\varepsilon \phi_2 dx, \\ \text{for any } \phi \in V. \end{cases} \quad (12)$$

The existence and uniqueness of the weak solution of (12) were established by Donato and Le Nguyen (2015) using the Minty-Browder Theorem.

Lemma 7. Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a solution to (12). Then for any $\varepsilon > 0$ and $k > 0$, there exists a $C > 0$, independent of ε and k , such that

$$\|T_k(u^\varepsilon)\|_V^2 \leq Ck \quad (13)$$

and

$$\int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \leq Ck. \quad (14)$$

Proof. Taking $\phi_i = T_k(u_i^\varepsilon)$ as a test function in (12), we obtain

$$\begin{aligned} & \int_{\Omega_1} A(x) \nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2} A(x) \nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon) dx \\ & + \int_{\Omega_1} g_1(x, u_1^\varepsilon) T_k(u_1^\varepsilon) dx + \int_{\Omega_2} g_2(x, u_2^\varepsilon) T_k(u_2^\varepsilon) dx \\ & + \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ & = \int_{\Omega_1} f^\varepsilon T_k(u_1^\varepsilon) dx + \int_{\Omega_2} f^\varepsilon T_k(u_2^\varepsilon) dx. \end{aligned} \quad (15)$$

By the ellipticity of A , for $i = 1, 2$,

$$\begin{aligned} \alpha \|\nabla T_k(u_i^\varepsilon)\|_{L^2(\Omega_i)}^2 &\leq \int_{\Omega_i} A(x) \nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) dx \\ &= \int_{\Omega_i} A(x) \nabla u_i^\varepsilon \nabla T_k(u_i^\varepsilon) dx \end{aligned} \quad (16)$$

Note that the integrals involving g_1 and g_2 are nonnegative. Indeed, since g_1 and g_2 satisfy Assumption $\mathcal{H}_1$ and T_k is nondecreasing with $T_k(0) = 0$, it follows then that for $i = 1, 2$,

$$\int_{\Omega_i} g_i(x, u_i) T_k(u_i) dx \geq 0. \quad (17)$$

For the integral over Γ , since h satisfies Assumption $\mathcal{H}_3$, from (4) of Remark 4, we have

$$h_0(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))(u_1^\varepsilon - u_2^\varepsilon) \leq h(x, u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)). \quad (18)$$

It then follows from (2) of Remark 4 that

$$\begin{aligned} \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma &\geq h_0 \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ &\geq h_0 \int_{\Gamma} (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))^2 d\sigma. \end{aligned} \quad (19)$$

For the integral on the right-hand side of (15), since $f^\varepsilon \rightarrow f$ converges strongly in $L^1(\Omega)$, it holds that there is some $M > 0$, independent of ε , such that

$$\|f^\varepsilon\|_{L^1(\Omega)} \leq M, \quad \forall \varepsilon > 0.$$

Then, since $|T_k| \leq k$,

$$\int_{\Omega_1} f^\varepsilon T_k(u_1^\varepsilon) dx + \int_{\Omega_2} f^\varepsilon T_k(u_2^\varepsilon) dx \leq \|f^\varepsilon\|_{L^1(\Omega)} k \leq Mk. \quad (20)$$

Inequalities (16) to (20), when combined with (15), imply that

$$\begin{aligned} \min\{\alpha, h_0\} \|T_k(u^\varepsilon)\|_V^2 &\leq \alpha \|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 + \alpha \|\nabla T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 \\ &+ \int_{\Gamma} (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) h(x, u_1^\varepsilon - u_2^\varepsilon) d\sigma \\ &\leq \alpha \|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 + \alpha \|\nabla T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 \\ &+ h_0 \|T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)\|_{L^2(\Gamma)}^2 \\ &\leq \|f^\varepsilon\|_{L^1(\Omega)} k \\ &\leq Mk. \end{aligned}$$

Since all the terms in the inequality above are nonnegative, we have (13) and (14). ■

The arguments used in the proof of the following lemma are analogous to those in Fulgencio and Guibé (2021a) and Guibé and Oropeza (2017).

Lemma 8. Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a solution to (12). Then there exists a measurable function $u = (u_1, u_2)$ in $\Omega \setminus \Gamma$ that is finite a.e., with

$$T_k(u) \in V \quad \text{for all } k \in \mathbb{N} \quad (21)$$

such that, up to a subsequence (still denoted by $\{\varepsilon\}$), the following convergences hold as ε tends to 0:

$$\begin{cases} u_i^\varepsilon \rightarrow u_i & \text{a.e. in } \Omega_i, \\ T_k(u_i^\varepsilon) \rightarrow T_k(u_i) & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ \gamma_i(u_i^\varepsilon) \rightarrow \gamma_i(u_i) & \text{a.e. on } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \rightarrow \gamma_i(T_k(u_i)) & \text{strongly in } L^2(\Gamma), \text{ a.e. in } \Gamma, \\ \nabla T_k(u_i^\varepsilon) \rightarrow \nabla T_k(u_i) & \text{weakly in } (L^2(\Omega_i))^N. \end{cases} \quad (22)$$

The following lemma is essential for passing to the limit of the integrals in the proof of the existence result (see the proof of Theorem 10).

Lemma 9. Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a weak solution to (12) and let $\{\varepsilon\}$ be the subsequence in Lemma 8. Then,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left(\int_{\Omega \setminus \Gamma} A(x) \nabla T_n(u^\varepsilon) \cdot \nabla T_n(u^\varepsilon) dx \right) &= 0 \\ \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left(\int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) &= 0. \end{aligned} \quad (23)$$

Proof. By Lemma 8, there is a subsequence, still denoted by ε , such that $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ satisfies (22). Using $\phi_i = \frac{1}{n} T_n(u_i^\varepsilon)$ as a test function in (12), we obtain

$$\begin{aligned} & \frac{1}{n} \int_{\Omega_1} A(x) \nabla u_1^\varepsilon \nabla T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} A(x) \nabla u_2^\varepsilon \nabla T_n(u_2^\varepsilon) dx \\ & + \frac{1}{n} \int_{\Omega_1} g_1(x, u_1^\varepsilon) T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} g_2(x, u_2^\varepsilon) T_n(u_2^\varepsilon) dx \\ & + \frac{1}{n} \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \\ & = \frac{1}{n} \int_{\Omega_1} f^\varepsilon T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} f^\varepsilon T_n(u_2^\varepsilon) dx. \end{aligned}$$

By (12) and (18), we have

$$\begin{aligned} & \frac{1}{n} \left(\int_{\Omega_1} A(x) \nabla T_n(u_1^\varepsilon) \nabla T_n(u_1^\varepsilon) dx + \int_{\Omega_2} A(x) \nabla T_n(u_2^\varepsilon) \nabla T_n(u_2^\varepsilon) dx \right. \\ & \left. + h_0 \int_{\Gamma} (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon))(u_1^\varepsilon - u_2^\varepsilon) d\sigma \right) \\ & \leq \frac{1}{n} \int_{\Omega_1} f^\varepsilon T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} f^\varepsilon T_n(u_2^\varepsilon) dx. \end{aligned} \quad (24)$$

Since $f^\varepsilon \rightarrow f$ strongly in $L^1(\Omega)$ and $T_n(u_i^\varepsilon) \rightarrow T_n(u_i)$ weakly-* in $L^\infty(\Omega_i)$, for $i = 1, 2$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i} f^\varepsilon T_n(u_i^\varepsilon) dx = \int_{\Omega_i} f T_n(u_i) dx.$$

Moreover, since for $i = 1, 2$, u_i is finite a.e.,

$$\frac{1}{n} T_n(u_i) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and since $|T_n(u_i)| \leq n$,

$$\left| \frac{1}{n} f T_n(u_i) \right| \leq |f| \in L^1(\Omega_i),$$

by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega_i} f T_n(u_i) dx = 0.$$

Taking the limit as $\varepsilon \rightarrow 0$ and then the limit $n \rightarrow \infty$ of both sides of (24), we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left(\int_{\Omega_1} A(x) \nabla T_n(u_1^\varepsilon) \nabla T_n(u_1^\varepsilon) dx + \int_{\Omega_2} A(x) \nabla T_n(u_2^\varepsilon) \nabla T_n(u_2^\varepsilon) dx \right. \\ & \left. + h_0 \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) \\ & \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \int_{\Omega_1} f T_n(u_1) dx + \frac{1}{n} \int_{\Omega_2} f T_n(u_2) dx \right) = 0. \end{aligned}$$

Since all integrals above are nonnegative, this proves (23). ■

Theorem 10. Let $\{u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)\}$ be a subsequence as in Lemma 8 with $u_i^\varepsilon \rightarrow u_i$ a.e. on Ω_i . Then $u = (u_1, u_2)$ is a renormalized solution of (1).

Proof. We show here that the limit $u = (u_1, u_2)$ of the subsequence $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a renormalized solution of (1) in the sense of Definition 5, that is, $u = (u_1, u_2)$ must satisfy (5)-(9).

Note that by (21) of Lemma 8, we have (5). For (6), by Fatou's lemma and inequalities (14) and (18),

$$\begin{aligned} \int_{\Gamma} (u_1 - u_2)(T_k(u_1) - T_k(u_2)) d\sigma &\leq \liminf_{\varepsilon \rightarrow 0} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{h_0} \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ &\leq \frac{Mk}{h_0}. \end{aligned}$$

This verifies (6). To prove that u satisfies (7) and (8), note that by lower semicontinuity of the weak convergence,

$$\begin{aligned} \frac{1}{n} \int_{\{|u| < n\}} A(x) \nabla u \cdot \nabla u dx &= \frac{1}{n} \int_{\Omega \setminus \Gamma} A(x) \nabla T_n(u) \cdot \nabla T_n(u) dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega \setminus \Gamma} A(x) \nabla T_n(u^\varepsilon) \cdot \nabla T_n(u^\varepsilon) dx. \end{aligned} \quad (25)$$

Moreover, by Fatou's lemma

$$\begin{aligned} & \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) d\sigma \\ & \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma. \end{aligned} \quad (26)$$

Then, conditions (7) and (8) follow from (23) of Lemma 9, (25) and (26).

Next, we show that u satisfies (9). Let $\psi_1, \psi_2 \in C^1(\mathbb{R})$ with compact support and let $k > 0$ such that

$$\text{supp } \psi_i \subseteq [-k, k] \quad (27)$$

and let $n \geq k$. Recall the function S_n defined in (10). Observe that S_n satisfies the following properties:

$$|S_n(r)| \leq 1 \quad \text{and} \quad |S'_n(r)| \leq \frac{1}{n}, \quad \text{for a.e. } r \in \mathbb{R}, \quad (28)$$

and

$$|S_n(s) - S_n(t)| \leq \frac{1}{n} |s - t|, \quad \text{for any } s, t \in \mathbb{R}. \quad (29)$$

Note first that

$$S_n(u_i^\varepsilon) = S_n(T_{2n}(u_i^\varepsilon)) \in H^1(\Omega_i) \cap L^\infty(\Omega_i), \quad i = 1, 2.$$

Let $\phi = (\phi_1, \phi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$, and set

$$v_i = \phi_i \psi_i(u_i) S_n(u_i^\varepsilon).$$

Using $v = (v_1, v_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ as a test function in (12), we have

$$I_{11} + I_{12} + I_{21} + I_{22} + I_{31} + I_{32} + I_{41} + I_{42} + I_5 = I_{61} + I_{62}, \quad (30)$$

where

$$\begin{aligned} I_{1i} &= \int_{\Omega_i} A(x) \nabla u_i^\varepsilon \cdot \nabla \phi_i \psi_i(u_i) S_n(u_i^\varepsilon) dx \\ I_{2i} &= \int_{\Omega_i} A(x) \nabla u_i^\varepsilon \cdot \nabla u_i \phi_i \psi_i'(u_i) S_n(u_i^\varepsilon) dx \\ I_{3i} &= \int_{\Omega_i} A(x) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon \phi_i \psi_i(u_i) S'_n(u_i^\varepsilon) dx \\ I_{4i} &= \int_{\Omega_i} g_i(x, u_i^\varepsilon) \phi_i \psi_i(u_i) S_n(u_i^\varepsilon) dx \\ I_5 &= \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) (\phi_1 \psi_1(u_1) S_n(u_1^\varepsilon) - \phi_2 \psi_2(u_2) S_n(u_2^\varepsilon)) d\sigma \\ I_{6i} &= \int_{\Omega_i} f v_i \psi_i(u_i) S_n(u_i^\varepsilon) dx. \end{aligned}$$

We evaluate the limits of integrals above as $\varepsilon \rightarrow 0$.

Note that since $\text{supp } S_n = [-2n, 2n]$, we have

$S_n(u_i^\varepsilon) \nabla u_i^\varepsilon = S_n(u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon)$.
 Moreover, by the boundedness of ψ and by (A2),
 $\psi_i(u_i) \nabla \phi \in L^2(\Omega_i)$ and $A(x) \in L^\infty(\Omega)$.
 By (22)₁ of Lemma 8 and (28), and the continuity of S_n ,
 $S_n(u_i^\varepsilon) \rightarrow S_n(u_i)$ weakly-* in $L^\infty(\Omega_i)$. (31)

It follows from (22) of Lemma 8 that

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} I_{1i} &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_i} A(x) \nabla T_{2n}(u_i^\varepsilon) \cdot \nabla \phi_i \psi_i(u_i) S_n(u_i^\varepsilon) dx \\
 &= \int_{\Omega_i} A(x) \nabla T_{2n}(u_i) \cdot \nabla \phi_i \psi_i(u_i) S_n(u_i) dx \\
 &= \int_{\Omega_i} A(x) \nabla u_i \cdot \nabla \phi_i \psi_i(u_i) S_n(u_i) dx.
 \end{aligned}$$

By (27), we may take n as large as possible so that $S_n(u_i) \psi_i(u_i) = \psi_i(u_i)$ and so,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{1i} = \int_{\Omega_i} \psi_i(u_i) A(x) \nabla u_i \cdot \nabla \phi_i dx. \quad (32)$$

For the limit of I_{2i} , by (27),

$$\nabla u_i \phi_i \psi_i'(u_i) = \nabla T_k(u_i) \phi_i \psi_i'(u_i) \in L^2(\Omega_i).$$

It follows from (31) and the fact that $n \geq k$ that

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} I_{2i} &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_i} A(x) \nabla T_k(u_i^\varepsilon) \cdot \nabla u_i \phi_i \psi_i'(u_i) S_n(u_i^\varepsilon) dx \\
 &= \int_{\Omega_i} A(x) \nabla T_k(u_i) \cdot \nabla u_i \phi_i \psi_i'(u_i) S_n(u_i) dx \\
 &= \int_{\Omega_i} A(x) \nabla u_i \cdot \nabla u_i \phi_i \psi_i'(u_i) S_n(u_i) dx.
 \end{aligned}$$

Using the fact that $\psi_i'(u_i) S_n(u_i) = \psi_i'(u_i)$ from (27),

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{2i} = \int_{\Omega_i} \phi_i \psi_i'(u_i) A(x) \nabla u_i \cdot \nabla u_i dx. \quad (33)$$

For I_{3i} , by (28) and (23),

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |I_{3i}| &\leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega_i} A(x) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon \phi_i \psi_i(u_i) S_n'(u_i^\varepsilon) dx \right| \\
 &\leq 2 \|\phi_i\|_{L^\infty(\Omega_i)} \|\psi_i\|_{L^\infty(\mathbb{R})} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{2n} \int_{|u_i| < 2n} A(x) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon dx \\
 &= 0.
 \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{3i} = 0. \quad (34)$$

For I_{4i} , observe that

$$\int_{\Omega_i} g_i(x, u_i^\varepsilon) \phi_i \psi_i(u_i) S_n(u_i^\varepsilon) dx = \int_{\Omega_i} g_i(x, T_{2n}(u_i^\varepsilon)) \phi_i \psi_i(u_i) S_n(u_i^\varepsilon) dx.$$

Note that by Remark 2,

$$|S_n(u_i^\varepsilon) g(x, T_{2n}(u_i^\varepsilon))| \leq B_g |T_{2n}(u_i^\varepsilon)| \leq 2n B_g,$$

and hence, by (22) of Lemma 8 and (31),

$$\rightarrow S_n(u_i) g(x, T_{2n}(u_i)) \text{ weakly-* in } L^\infty(\Omega_i), \text{ for a.e. } x \in \Omega.$$

This gives

$$\lim_{\varepsilon \rightarrow 0} I_{4i} = \int_{\Omega_i} g_i(x, T_{2n}(u_i)) \phi_i \psi_i(u_i) S_n(u_i) dx.$$

Since $g_i(x, T_{2n}(u_i)) S_n(u_i) \psi_i(u_i) = g_i(x, u_i) \psi_i(u_i)$ and $n \geq k$, we obtain

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{4i} = \int_{\Omega_i} \psi_i(u_i) g_i(x, u_i) \phi_i dx. \quad (35)$$

For I_5 , we may write

$$I_5 = I_{51} + I_{52} + I_{53} - I_{54} \quad (36)$$

where

$$I_{51} = \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_1 \psi_1(u_1) S_{2n}(u_1^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) d\sigma,$$

$$I_{52} = \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_1 \psi_1(u_1) S_n(u_2^\varepsilon) S_{2n}(u_1^\varepsilon) d\sigma,$$

$$I_{53} = \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_2 \psi_2(u_2) S_{2n}(u_2^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) d\sigma,$$

$$I_{54} = \int_{\Gamma} h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_2 \psi_2(u_2) S_n(u_1^\varepsilon) S_{2n}(u_2^\varepsilon) d\sigma.$$

By (29) and noting that $S_n(u_i^\varepsilon) = S_n(T_{2n}(u_i^\varepsilon))$, we have

$$|S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)| \leq \frac{1}{n} |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)|. \quad (37)$$

It that

$$\begin{aligned}
 |I_{51}| &\leq h_1 \|\phi_1\|_{L^\infty(\Gamma)} \|\psi_1\|_{L^\infty(\mathbb{R})} \\
 &\quad \|\psi_1\|_{L^\infty(\mathbb{R})} \frac{1}{n} \int_{\Gamma} |u_1^\varepsilon - u_2^\varepsilon| |T_n(u_1^\varepsilon) \\
 &\quad - T_n(u_2^\varepsilon)| d\sigma.
 \end{aligned}$$

By (23),

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{51} = 0. \quad (38)$$

Similarly,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{53} = 0. \quad (39)$$

For I_{52} , note that

$$\begin{aligned}
 &|h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_1 \psi_1(u_1) S_n(u_2^\varepsilon) S_{2n}(u_1^\varepsilon)| \\
 &= |h(x, T_{4n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)) \phi_1 \psi_1(u_1) S_n(u_2^\varepsilon) S_{2n}(u_1^\varepsilon)| \\
 &\leq h_1 \|\phi_1\|_{L^\infty(\Gamma)} \|\psi_1\|_{L^\infty(\mathbb{R})} |T_{4n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)| \\
 &\leq h_1 \|\phi_1\|_{L^\infty(\Gamma)} \|\psi_1\|_{L^\infty(\mathbb{R})} (6n).
 \end{aligned}$$

Since

$$\begin{aligned}
 &h(x, u_1^\varepsilon - u_2^\varepsilon) \phi_1 \psi_1(u_1) S_n(u_2^\varepsilon) S_{2n}(u_1^\varepsilon) \\
 &\rightarrow h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_2) S_{2n}(u_1)
 \end{aligned}$$

a.e. on Γ as $\varepsilon \rightarrow 0$, it follows from Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \rightarrow 0} I_{52} = \int_{\Gamma} h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_2) S_{2n}(u_1) d\sigma.$$

Recall that $\text{supp} \psi_i \subseteq [-k, k]$ from (27) and $n \geq k$, so that

$$\begin{aligned}
 &h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_2) S_{2n}(u_1) \\
 &= h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_2).
 \end{aligned}$$

Since $S_n(u_2)$ converges to 1 a.e. on Γ and

$$\begin{aligned}
 &|h(x, u_1 - u_2) \phi_1 \psi_1(u_1) S_n(u_2) S_{2n}(u_1)| \\
 &\leq h_1 |T_{2n}(u_1) - T_{2n}(u_2)| \phi_1 \psi_1(u_1) \\
 &\leq 4n h_1 \|\phi_1\|_{L^\infty(\Gamma)} \|\psi_1\|_{L^\infty(\mathbb{R})}.
 \end{aligned}$$

It follows from Lebesgue Dominated Convergence Theorem that

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{52} = \int_{\Gamma} h(x, u_1 - u_2) \phi_1 \psi_1(u_1) d\sigma. \quad (40)$$

It can also be shown with the same argument that

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{54} = \int_{\Gamma} h(x, u_1 - u_2) \phi_2 \psi_2(u_2) d\sigma. \quad (41)$$

Hence, combining (36), (38), (39), (40), and (41) yields

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_5 = \int_{\Gamma} (\phi_1 \psi_1(u_1) - \phi_2 \psi_2(u_2)) h(x, u_1 - u_2) d\sigma. \quad (42)$$

Lastly, for I_{6i} , by (31),

$$\lim_{\varepsilon \rightarrow 0} I_{6i} = \int_{\Omega_i} f v_i \psi_i(u_i) S_n(u_i) dx.$$

Noting (27), we obtain

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{6i} = \int_{\Omega_i} f v_i \psi_i(u_i) dx.$$

Combining (32), (33), (34), (35), (42) and (43) into (30) gives us the renormalized formulation (9). This completes the. ■

We now turn to uniqueness. We first prove the following lemma showing that the solution and its trace are L^1 -functions. This regularity is essential for passing to the limit in some of the integrals.

Lemma 11. *Let u be a renormalized solution of (1). Then $u_i \in W^{1,q}(\Omega_i)$, $i = 1, 2$, where $1 \leq q < \frac{N}{N-1}$. In particular, $u_i \in L^1(\Omega_i)$ and $\gamma_i(u_i) \in L^1(\Gamma)$ for $i = 1, 2$, where γ_i is the trace operator defined in Proposition 5.*

Proof. We follow the arguments used in Boccardo and Gallouët (1989). We begin by showing that

$$\int_{D_{i,n}} |\nabla u_i|^2 dx \leq C,$$

where

$$D_{i,n} = \{x \in \Omega: n \leq |u_i(x)| < n+1\}.$$

Fix $n, k \in \mathbb{N}$ such that $k \geq n+1$. Define θ_n as

$$\theta_n(s) = \begin{cases} 1, & \text{if } s > n+1 \\ s-n, & \text{if } n \leq s \leq n+1 \\ 0, & \text{if } -n \leq s \leq n \\ s+n, & \text{if } -n-1 < s < -n \\ -1, & \text{if } s < -n-1. \end{cases}$$

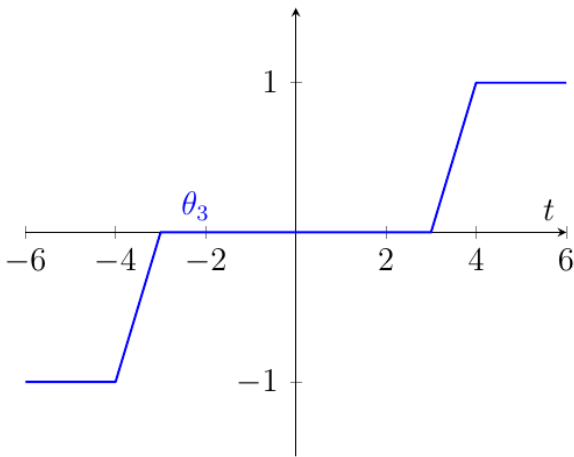


Figure 4: The graph of the function θ_3 .

Then $\theta_n(u) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$. Letting $\phi_i = \theta_n(u_i)$ and $\psi_i = S_k$ (as defined in (10)) in (9), we obtain

$$A_1 + A_2 + B_1 + B_2 + C_1 + C_2 + D = E_1 + E_2, \quad (44)$$

where

$$A_i = \int_{\Omega_i} S_k(u_i) \theta'_n(u_i) A(x) \nabla u_i \nabla u_i dx$$

$$B_i = \int_{\Omega_i} \theta_n(u_i) S'_k(u_i) A(x) \nabla u_i \nabla u_i dx$$

$$C_i = \int_{\Omega_i} S_k(u_i) \theta_n(u_i) g_i(x, u_i) dx$$

$$D = \int_{\Gamma} h(x, u_1 - u_2) (S_k(u_1) \theta_n(u_1) - S_k(u_1) \psi_n(u_2)) d\sigma$$

$$E_i = \int_{\Omega_i} f S_k(u_i) \theta_n(u_i) dx.$$

We compute the limit of each term in (44) as $k \rightarrow +\infty$.

For A_i , since $k \geq n+1$, we have the following inequality:

$$|S_k(u_i) \theta'_n(u_i) A(x) \nabla u_i \nabla u_i| \leq |A(x) \nabla T_{n+1}(u_i) \nabla T_{n+1}(u_i)| \in L^1(\Omega_i).$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k \rightarrow +\infty} \int_{\Omega_i} S_k(u_i) \theta'_n(u_i) A(x) \nabla u_i \nabla u_i dx = \int_{\Omega_i} \theta'_n(u_i) A(x) \nabla u_i \nabla u_i dx.$$

Thus, by the ellipticity of A ,

$$\lim_{k \rightarrow +\infty} A_i = \int_{\{n < |u_i| \leq n+1\}} A(x) \nabla u_i \nabla u_i dx \geq \alpha \int_{\{n < |u_i| \leq n+1\}} |\nabla u_i|^2 dx. \quad (45)$$

For B_i , by (28) and from the definition of $|\theta_n| \leq 1$, we have

$$|\theta_n(u_i) S'_k(u_i) A(x) \nabla u_i \nabla u_i| \leq \frac{1}{k} |A(x) \nabla T_k(u_i) \nabla T_k(u_i)| \in L^1(\Omega_i).$$

Thus,

$$\lim_{k \rightarrow \infty} |B_i| \leq \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Omega_i} |A(x) \nabla T_k(u_i) \nabla T_k(u_i)| dx = 0. \quad (46)$$

Now, for C_i , note that when $s \leq 0$, $\theta_n(s) \leq 0$ and $g(x, s) \leq 0$, and if $s \geq 0$, $\theta_n(s) \geq 0$ and $g(x, s) \geq 0$. Hence, $\theta_n(s) g(x, s) \geq 0$, for any $s \in \mathbb{R}$. Since S_k is nonnegative,

$$C_i \geq 0. \quad (47)$$

For D , consider the following decomposition

$$D = \int_{\Gamma} h(x, u_1 - u_2) S_k(u_2) (\theta_n(u_1) - \theta_n(u_2)) d\sigma + \int_{\Gamma} h(x, u_1 - u_2) (S_k(u_1) - S_k(u_2)) \theta_n(u_1) d\sigma.$$

Since h satisfies Assumption $\mathcal{H}_3$, S_k is nonnegative and θ_n is nondecreasing, we may apply (4) of Remark 4 to get

$$\begin{aligned} & h(x, u_1 - u_2) S_k(u_2) (\theta_n(u_1) - \theta_n(u_2)) \\ &= (u_1 - u_2) h(x, u_1 - u_2) S_k(u_2) \frac{\theta_n(u_1) - \theta_n(u_2)}{u_1 - u_2} \\ &\geq C(u_1 - u_2)^2 S_k(u_2) \frac{\theta_n(u_1) - \theta_n(u_2)}{u_1 - u_2} \geq 0 \end{aligned}$$

Then

$$D \geq \int_{\Gamma} h(x, u_1 - u_2) (S_k(u_1) - S_k(u_2)) \theta_n(u_1) d\sigma.$$

Using inequality (37),

$$|S_k(u_1) - S_k(u_2)| \leq \frac{1}{k} |T_{2k}(u_1) - T_{2k}(u_2)|.$$

Then, Remark 2 implies that for all k ,

$$\begin{aligned} & \int_{\Gamma} |h(x, u_1 - u_2) (S_k(u_1) - S_k(u_2)) \theta_n(u_1)| d\sigma \\ &\leq \frac{B_h}{k} \int_{\Gamma} |(u_1 - u_2) (T_{2k}(u_1) - T_{2k}(u_2))| d\sigma. \end{aligned}$$

Since u is a renormalized solution of (1), by (8), we have

$$\lim_{k \rightarrow \infty} \int_{\Gamma} |h(x, u_1 - u_2) (S_k(u_1) - S_k(u_2)) \theta_n(u_1)| d\sigma = 0.$$

Thus,

$$\liminf_{k \rightarrow \infty} D \geq 0. \quad (48)$$

Observe that $|f S_k(u_i) \theta_n(u_i)| \leq |f| \in L^1(\Omega)$ by the boundedness of S_k and θ_k above by 1, and $S_k(u) \rightarrow 1$ a.e. on Ω . By Lebesgue dominated convergence theorem,

$$\lim_{k \rightarrow \infty} |E_i| \leq \int_{\Omega} |f| dx. \quad (49)$$

Altogether, (44), (45), (46), (47), (48) and (49) imply that

$$\int_{D_{i,n}} |\nabla u_i|^2 dx \leq \frac{1}{\alpha} \|f\|_{L^1(\Omega)}.$$

Applying Lemma 1 of Boccardo and Gallouët (1989), we conclude that $u_i \in W^{1,q}(\Omega_i)$ for $1 \leq q < \frac{N}{N-1}$. ■

Theorem 12. If either g_2 or h is strictly increasing, then the renormalized solution of (1) is unique.

Proof. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two renormalized solutions of (1). Let $n \in \mathbb{N}$ and $k > 0$. Note that

$$S_n(u)S_n(v)T_k(T_{2n}(u) - T_{2n}(v)) \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Setting $\psi_i = S_n$ and $\phi = S_n(v)(T_k(T_{2n}(u) - T_{2n}(v)))$ in the renormalized formulation (9) of Definition 5 for u , we obtain $I_{11} + I_{12} + I_{21} + I_{22} + I_{31} + I_{32} + I_{41} + I_{42} + I_5 = I_{61} + I_{62}$, (50) where for $i = 1, 2$,

$$\begin{aligned} I_{1i} &= \int_{\Omega_i} S_n(u_i)S_n'(v_i)T_k(u_i - v_i)A(x)\nabla u_i \cdot \nabla v_i dx \\ I_{2i} &= \int_{\Omega_i} S_n(u_i)S_n(v_i)\chi_{\{|u_i - v_i| \leq k\}}(x)A(x)\nabla u_i \cdot (\nabla u_i - \nabla v_i) dx \\ I_{3i} &= \int_{\Omega_i} S_n(v_i)S_n'(u_i)T_k(u_i - v_i)A(x)\nabla u_i \nabla u_i dx \\ I_{4i} &= \int_{\Omega_i} S_n(u_i)S_n(v_i)g_i(x, u_i)(T_k(u_i - v_i)) dx \\ I_5 &= \int_{\Gamma} h(x, u_1 - u_2)(S_n(u_1)S_n(v_1)T_k(u_1 - v_1) - S_n(u_2)S_n(v_2)T_k(u_2 - v_2)) d\sigma \\ I_{6i} &= \int_{\Omega_i} f S_n(u_i)S_n(v_i)T_k(u_i - v_i) dx. \end{aligned}$$

Similarly, setting $\phi_i = S_n$ and $\phi = S_n(u)(T_k(T_{2n}(u) - T_{2n}(v)))$ in the renormalized formulation (9) of Definition 5 for v gives us $J_{11} + J_{12} + J_{21} + J_{22} + J_{31} + J_{32} + J_{41} + J_{42} + J_5 = J_{61} + J_{62}$, (51) where for $i = 1, 2$,

$$\begin{aligned} J_{1i} &= \int_{\Omega_i} S_n(v_i)S_n'(u_i)T_k(u_i - v_i)A(x)\nabla v_i \cdot \nabla u_i dx \\ J_{2i} &= \int_{\Omega_i} S_n(v_i)S_n(u_i)\chi_{\{|u_i - v_i| \leq k\}}(x)A(x)\nabla v_i \cdot (\nabla u_i - \nabla v_i) dx \\ J_{3i} &= \int_{\Omega_i} S_n(u_i)S_n'(v_i)T_k(u_i - v_i)A(x)\nabla v_i \nabla v_i dx \\ J_{4i} &= \int_{\Omega_i} S_n(u_i)S_n(v_i)g_i(x, v_i)(T_k(u_i - v_i)) dx \\ J_5 &= \int_{\Gamma} h(x, v_1 - v_2)(S_n(u_1)S_n(v_1)T_k(u_1 - v_1) - S_n(u_2)S_n(v_2)T_k(u_2 - v_2)) d\sigma \\ J_{6i} &= \int_{\Omega_i} f S_n(u_i)S_n(v_i)T_k(u_i - v_i) dx. \end{aligned}$$

Subtracting (51) from (50), we consider the limits of the differences of the corresponding terms as n tends to infinity with k fixed.

For I_{1i} and J_{1i} , using that matrix field belongs to $M(\alpha, \beta, \Omega)$, that $|T_k| \leq k$, and (28), to get

$$|I_{1i}|^2 \leq \left(\frac{k\beta}{n}\right)^2 \left(\int_{\Lambda_i} |\nabla u_i| \cdot |\nabla v_i| dx\right)^2$$

and

$$|J_{1i}|^2 \leq \left(\frac{k\beta}{n}\right)^2 \left(\int_{\Lambda_i} |\nabla u_i| \cdot |\nabla v_i| dx\right)^2,$$

where $\Lambda_i = \{|u_i| < 2n\} \cap \{|v_i| < 2n\}$, $i = 1, 2$. Now, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left(\frac{k\beta}{n}\right)^2 \left(\int_{\Lambda_i} |\nabla u_i| \cdot |\nabla v_i| dx\right)^2 \\ &\leq \left(\frac{k\beta}{n}\right)^2 \left(\int_{\Lambda_i} |\nabla u_i|^2 dx\right) \left(\int_{\Lambda_i} |\nabla v_i|^2 dx\right) \\ &\leq \left(\frac{k\beta}{n}\right)^2 \left(\frac{1}{\alpha} \int_{\Lambda_i} A(x)\nabla u_i \nabla u_i dx\right) \left(\frac{1}{\alpha} \int_{\Lambda_i} A(x)\nabla v_i \nabla v_i dx\right) \\ &= \left(\frac{2k\beta}{\alpha}\right)^2 \left(\frac{1}{2n} \int_{\{|u_i| < 2n\}} A(x)\nabla u_i \nabla u_i dx\right) \left(\frac{1}{2n} \int_{\{|v_i| < 2n\}} A(x)\nabla v_i \nabla v_i dx\right). \end{aligned}$$

By (7),

$$\lim_{n \rightarrow \infty} |I_{1i}| = 0 = \lim_{n \rightarrow \infty} |J_{1i}| = 0,$$

and hence

$$\lim_{n \rightarrow \infty} (I_{1i} - J_{1i}) = 0. \quad (52)$$

We estimate I_{3i} and J_{3i} in the same way. We use (28) and the fact that

$$S_n'(u_i)\nabla u_i = S_n'(u_i)\nabla T_n(u_i),$$

to obtain the estimates

$$\begin{aligned} |I_{3i}| &\leq \int_{\Omega_i} |S_n(v_i)S_n'(u_i)(T_k(u_i - v_i)A(x)\nabla T_{2n}(u_i)\nabla T_{2n}(u_i))| dx \\ &\leq \frac{k}{n} \int_{\{|u_i| \leq 2n\}} A(x)\nabla u_i \nabla u_i dx. \end{aligned}$$

Similarly,

$$|J_{3i}| \leq \frac{k}{n} \int_{\{|v_i| \leq 2n\}} A(x)\nabla v_i \nabla v_i dx.$$

Since u and v are renormalized solutions of (1), we apply (7) to get

$$\begin{cases} \lim_{n \rightarrow \infty} |I_{3i}| \leq 2k \cdot \lim_{n \rightarrow \infty} \frac{1}{2n} \int_{\{|u_i| \leq 2n\}} A(x)\nabla u_i \nabla u_i dx = 0 \\ \lim_{n \rightarrow \infty} |J_{3i}| \leq 2k \cdot \lim_{n \rightarrow \infty} \frac{1}{2n} \int_{\{|u_i| \leq 2n\}} A(x)\nabla v_i \nabla v_i dx = 0. \end{cases} \quad (53)$$

Next, observe that both $S_n(u_i)$ and $S_n(v_i)$ converge to 1 weakly-* in $L^\infty(\Omega_i)$. So,

$$\begin{aligned} &\lim_{n \rightarrow \infty} (I_{2i} - J_{2i}) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_i} S_n(v_i)S_n(u_i)\chi_{\{|u_i - v_i| \leq k\}}(x)A(x)(\nabla u_i - \nabla v_i) \cdot (\nabla u_i - \nabla v_i) dx \\ &= \int_{\Omega_i} \chi_{\{|u_i - v_i| \leq k\}}(x)A(x)(\nabla u_i - \nabla v_i) \cdot (\nabla u_i - \nabla v_i) dx. \end{aligned} \quad (54)$$

For $I_{4i} - J_{4i}$, note first that by Remark 2,

$|(g_i(x, u_i) - g_i(x, v_i))T_k(u_i - v_i)| \leq k|u_i - v_i| \in L^1(\Omega_i)$, where the last inclusion is by Lemma 11. Since $S_n(u_i)S_n(v_i)$ converges to 1 weakly-* in $L^\infty(\Omega_i)$,

$$\begin{aligned} &\lim_{n \rightarrow \infty} (I_{4i} - J_{4i}) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_i} S_n(u_i)S_n(v_i)(g_i(x, u_i) - g_i(x, v_i))(T_k(u_i - v_i)) dx \\ &= \int_{\Omega_i} (g_i(x, u_i) - g_i(x, v_i))(T_k(u_i - v_i)) dx. \end{aligned} \quad (55)$$

For I_5 , we have

$$I_5 = \int_{\Gamma} h(x, u_1 - u_2)(S_n(u_1)S_n(v_1)T_k(u_1 - v_1) - S_n(u_2)S_n(v_2)T_k(u_2 - v_2)) d\sigma.$$

By Remark 2,

$$|h(x, u_1 - u_2)| \leq h_1|u_1 - u_2|.$$

Then Lemma 11 implies that $h(x, u_1 - u_2)$ is integrable on Γ . Now that

$$\begin{aligned} &|(S_n(u_1)S_n(v_1)(T_k(u_1 - v_1)) - S_n(u_2)S_n(v_2)(T_k(u_2 - v_2)))h(x, u_1 - u_2)| \\ &\leq |T_k(u_1 - v_1)h(x, u_1 - u_2)| + |T_k(u_2 - v_2)h(x, u_1 - u_2)| \\ &\leq 2k|h(x, u_1 - u_2)| \in L^1(\Gamma). \end{aligned}$$

Lebesgue dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} I_5 = \int_{\Gamma} (T_k(u_1 - v_1) - T_k(u_2 - v_2))h(x, u_1 - u_2) d\sigma. \quad (56)$$

Similarly,

$$\lim_{n \rightarrow \infty} J_5 = \int_{\Gamma} (T_k(u_1 - v_1) - T_k(u_2 - v_2))h(x, v_1 - v_2) d\sigma. \quad (57)$$

Since h is nondecreasing on the second variable, inequality (3) of Remark 4 implies that

$$(T_k(u_1 - v_1) - T_k(u_2 - v_2))(h(x, u_1 - u_2) - h(x, v_1 - v_2)) \geq 0.$$

Lastly, $I_{6i} = J_{6i}$ and so, combining (52), (53), (54), (55), (56) and (57), we obtain the limit of the difference of (50) and (51) as $n \rightarrow \infty$,

$$\begin{aligned}
& \sum_{i=1,2} \int_{\Omega_i} \chi_{\{|u_i - v_i| \leq k\}}(x) A(x) (\nabla u_i - \nabla v_i) \cdot (\nabla u_i - \nabla v_i) dx \\
& + \sum_{i=1,2} \int_{\Omega_i} (g_i(x, u_i) - g_i(x, v_i)) (T_k(u_i - v_i)) dx \\
& + \int_{\Gamma} (T_k(u_1 - v_1) - T_k(u_2 - v_2)) (h(x, u_1 - u_2) - h(x, v_1 - v_2)) d\sigma = 0,
\end{aligned} \quad (58)$$

for all $k > 0$. Since each of the integrals in (58) is non-negative, all of them are actually zero. Since g or h_2 is strictly increasing, we obtain $u_i = v_i$ a.e. on Ω_i , $i = 1, 2$. This also implies that $\gamma_i(u_i) = \gamma_i(v_i)$, $i = 1, 2$, a.e. on Γ . Thus, the renormalized formulation of (1) is unique. ■

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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